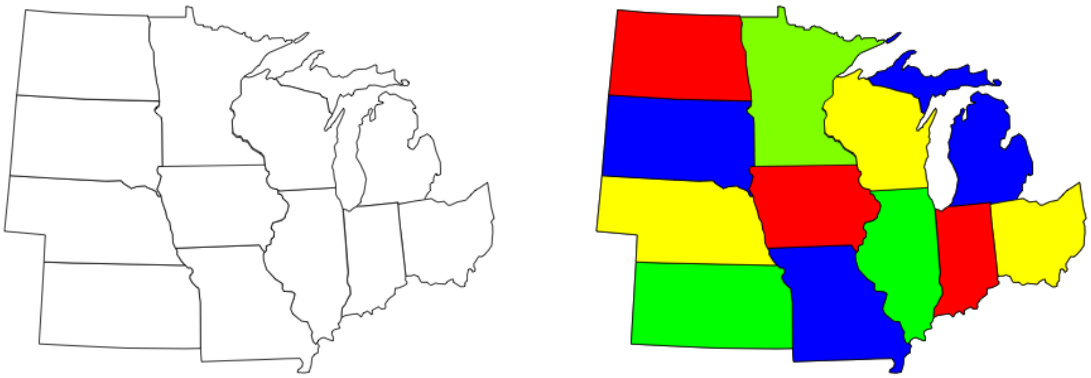
Chapter 5.1 Coloring Planar Graphs

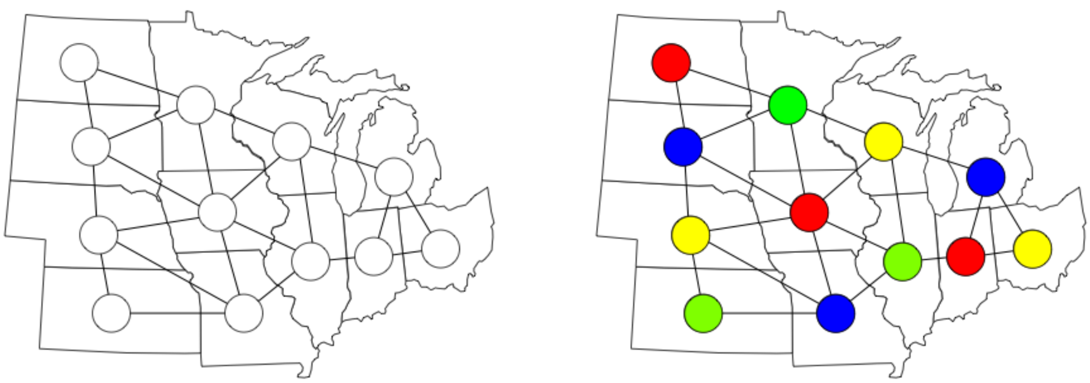
Problem: Color regions of the plane such that regions sharing border get different colors. Show that 4 colors is enough (if regions connected) for any set of regions.
(Restating: Is it true that every planar graph is 4-colorable? Answer is yes.)

1: Color the states in midwest by 4 colors with neighboring ones have distinct colors. Can you do 3 colors?



The problem can be turned into a graph problem by having a vertex for every region.

2: Translate your coloring to a coloring of the graph below.



Let G be a graph and C be a set of colors. A **coloring** is a mapping $c : V(G) \to C$ such that $c(u) \neq c(v)$ for all $uv \in E(G)$. Sometimes called **proper coloring**.

A graph G is k**-colorable** is there exists a (proper) coloring of G using k colors.

Chromatic number of G, denoted by $\chi(G)$ is the minimum k such that G is k-colorable.

3: Decide what is the chromatic number of C_k. (try $3 \leq k \leq 7$)

Solution: $\chi(C_4) = \chi(C_6) = 2$ and $\chi(C_3) = \chi(C_5) = \chi(C_7) = 3$.
For all even cycles, $\chi(C_{2k}) = 2$ and for odd cycles $\chi(C_{2k+1}) = 3$.

A set of vertices $X \subset V(G)$ are **independent** in a graph G if $G[X]$ has no edges.

Let c be a (proper) coloring of G. If V_{red} is the set of vertices colored red then V_{red} is an independent set.

Coloring G by k colors is a decomposition of $V(G)$ into in k independent sets. $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$.

4: Show that $\chi(G) = 2$ iff G is bipartite (and has at least one edge).

Solution: If G is 2-colorable, then each of the two color classes is an independent set and it creates the desired bipartition. If G is bipartite, color one class with color one and the other with color two. That creates the desired coloring.
Notice that G is bipartite iff it does not contain an odd cycle as a subgraph. Hence we get a characterization that G is 2-colorable iff G does not contain an odd cycle as a subgraph. No nice characterization is known for more than 2 colors.

5: What is $\chi(K_n)$?

Solution: $\chi(K_n) = n$ since everyone is connected with everyone. This gives an easy lower bound.

A **clique** in a graph G is a subgraph that is isomorphic to a complete graph.
The **clique number**, $\omega(G)$, is the order of the largest clique in G.

Recall $\Delta(G)$ is the maximum degree of a vertex in G.

6: Show that for every graph G holds $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$.

Solution: Vertices in a clique has to be colored by different colors. Hence $\omega(G) \leq \chi(G)$. For the other bound, color the vertices one by one. Everytime you are about to color a vertex, it has at most $\Delta(G)$ neighbors that are already colored. So it needs to avoid at most $\Delta(G)$ colors, but there are $\Delta(G) + 1$ colors available.

Theorem Every planar graph is 4-colorable.
A serious theorem. No simple proof known. Computer assisted.

7: Show that every planar graph is 6-colorable. Recall that $\delta(G) \leq 5$ if G is a planar graph.

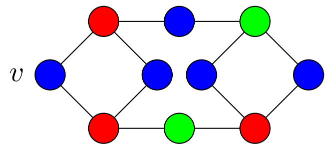
Solution: We use proof by induction. Let G be a planar graph on n vertices and graphs on $n-1$ vertices are 6-colorable. Let v be a vertex of G of degree 5. By induction, $G - v$ is 6-colorable by coloring c. Since v has only 5 neighbors, there is at least one if the 6 colors not used on neighbors of v and c can be extended to a 6-coloring of G.

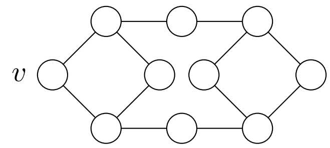
Theorem Every planar graph is 5-colorable.

We do the proof by induction on the number of vertices. We use Kempe chains.

Let c be a coloring of G. A **Kempe chain** in colors 1 and 2 is a maximal connected subgraph of G where all vertices are colored 1 and 2.

8: In the following graph, take a Kempe chain in colors red and blue that contains the vertex v and swap *red* and *blue* colors on it. Draw the result on the right.





9: Let $K \subseteq V(G)$ be a Kempe chain in G for a coloring c. Let c' be obtained from c by swapping colors 1 and 2 on K but nowhere else.

$$c'(v) = \begin{cases} 1 & \text{if } v \in K \text{ and } c(v) = 2 \\ 2 & \text{if } v \in K \text{ and } c(v) = 1 \\ c(v) & \text{otherwise} \end{cases}$$

Show that c' is a proper coloring.

Solution: Suppose that c' is not a proper coloring. Assume that there are two vertices u and v such that $c'(u) = c'(v) = a$. Since it only changes colors are 1 and 2, it must hold that $a \in \{1, 2\}$. If both $u, v \in K$, then they should have different colors. Same if both $u, v \notin K$. Hence we conclude $u \in K$ and $v \notin K$. This is a contradiction with the maximality of K.

Proof of the 5-color theorem. Let G be a plane graph on n vertices. Assume that all planar graphs on at most $n - 1$ vertices are 5-colorable.

10: Show that G is 5-colorable if it contains a vertex v of degree at most four.

Solution: If G contains a vertex v of degree at most 4, we consider a 5-coloring of $G - v$ and extend it to a 5-coloring of G since there are at most 4 colors used on the neighbors of v.

Let v be a vertex of degree 5 in G. By induction, there is a 5-coloring c of $G-v$. If the neighbors of v use at most 4 colors in c, the coloring c can be extended to v. Hence assume that the neighbors of v are colored by $1,2,3,4,5$ (in clockwise order in the drawing on G).

11: Use Kempe chain in colors 1 and 3 and another one in colors 2 and 4 to show that there exists a coloring c' of $G-v$ such that the neighbors of G have at most 4 colors.

Solution: Let v_i be a neighbor of v colored by color i. First we try to take the Kempe chain $K_{1,3}$ in colors 1 and 3 that contains v_1. If $K_{1,3}$ does not contain v_3, then switching colors on $K_{1,3}$ creates a coloring that can be extended to v by coloring v by 1. So assume that $v_3 \in K_{1,3}$. So we take a Kempe chain $K_{2,4}$ in colors 2 and 4 that contains v_2. Observe that by planarity, $v_4 \notin K_{2,4}$. Hence we can flip colors on $K_{2,4}$ and color v by color 2. This finishes the proof of 5-color theorem.

Theorem Grötzsch 1959
Every planar triangle-free graphs is 3-colorable.